Fourier's Transformation and Theorem

Math 5 [Ordinary Differential Equations] - Honors Project

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May 26, 2020

Abstract

The project explores Fourier's transformation and its derivations. Using the concepts learned in the differential equations class, the paper discusses and presents the mathematical differentiation of the Fourier theorem, as well as the necessary theorem known as Euler's Formula. The paper explores the concept of Fourier's Series and transforms to translate wave functions from an imaginary frequency space, to real space by the use of matrices multiplication resulted from the summation of trigonometric periodic functions. The paper discusses issues that the applications of the Fourier theorem has encountered, and presents the mathematical algorithms used to solve them (Cooley-Tukey, Butterfly Diagrams, and matrix multiplication). Furthermore, the paper explore the various applications of the theorem, and examines its uses in the modern technology world.

Keywords:

Euler's Formula, Frequcny Space, Fourier's Series, Fourier Transforms; Cooley-Tukey Algorithm; Fast Fourier Transformation.

1 Introduction and History

The first thought of combining periodic functions into a simply sum of oscillating motion was first recorded in the 3rd century BC, in Egyptian Mathematics. The next appearance of such an idea, was in 1805(2). **Joseph Fourier**, a French Mathematician who made significant contributions to the study of trigonometric series, had established the idea that any arbitrary (continuous) function on the interval [0,1] can be represented by a trigonometric series(2). Fourier's Series was originally used to solve the heat equation (9) (a Partial Differential Equation that describes heat flow in a medium) in a metal plate,

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \cdot c} \cdot \frac{\partial^2 T}{\partial x^2}$$

however, when presented to the French Academy in 1807, the series's true potential was discovered. The heat equations, as was presented in the 1800's, represented the spread of heat in a space (a medium) over time. Fourier modeled a complicated heat source as a superposition - a linear combination - of simple sine and cosine waves, which can be thought of as the summation of various wave functions. The summation, which based on calculus mathematics, can be represented as a series over a specified range [R - Period of the series], which allowed Fourier to create the following series to solve the heat equation

$$\sum_{n=0}^{N} (a_n \cdot cos(\frac{2\pi nt}{P}) + b_n \cdot sin(\frac{2\pi nt}{P}))$$

where like in simple harmonic motion, and oscillating motion,

Symbol	Meaning (Definition)
n	number of cycles
a and **b**	Fourier's Coefficients
P	Period[ic motion]

Hence, we can define the followings as well

Symbol	Meaning (Definition)	Relationship (Formula)
f	corresponding harmonic frequency	f = n / P
ω	Angular Frequency	$\omega = 2 \cdot \pi f = 2 \cdot \pi \frac{n}{P}$

Therefore, we can define **Fourier's Series** as:

$$F(t) = s_n(t) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cdot cos(\omega \cdot t)) + b_n \cdot sin(\omega \cdot t)) \qquad \text{(eqn. FS-1)}$$

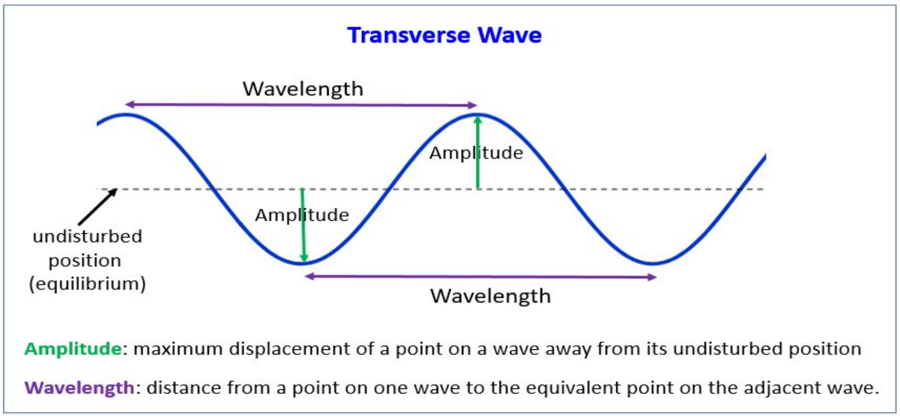
For the sake of mathematical symmetry, the series could also be defined as:

$$F(t) = s_n(t) = \sum_{n=-\infty}^{\infty} (a_n \cdot cos(\omega \cdot t)) + b_n \cdot sin(\omega \cdot t)) \qquad \text{(eqn. FS-2)}$$

Although originally the series was used to solve the famous heat equations, it could also be applied in various problems in math, classical and modern quantum mechanical physics, and computer science and voice recognition. With some modifications to Fourier's series, a mathematical tool known as **Fourier's Transforms** was developed. The tool allowed the conversion (transformation, translation of values from imaginary planes (spaces), to real ones. Practically, Fourier's series became a reliable mathematical tool in Engineering to approximate and calculate superposition of any sinusoidal wave.

2 Conceptually

The mathematical development of the **Fourier Transforms** is a strong prove that math is developed to solve real life problems, and not simple just "found". To understand the concept of Fourier series and transformation, the concept of a wave must be understood first. A wave is a "Disturbance in a medium that propagates of its volition" (7). A wave is defined mathematically by four major properties: Amplitude (A), Frequency (f), Period(T), and Wavelength (λ).



(7)

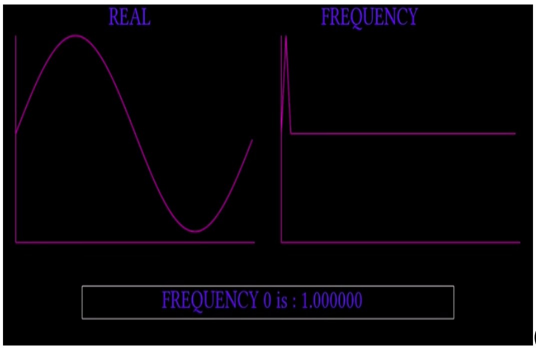
Each propagating wave can be identified by the following relationships(7):

Relationship	Formula
Cycle Period	T = $f^{-1} = 1/f$
wave speed	v = $\lambda \cdot f$
wave number	k = $2 \cdot \pi/\lambda$
Traverse Position	y(x, t) = A $\cdot \sin(kx - \omega t)$

By differentiating the traverse position equation of a traversal periodic wave both with respect to time, and horizontal displacement (x), we get the general **wave equation**(7):

$$\frac{\partial^2 y}{\partial t^2}(x,t) = v^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{(eqn. W)}$$

The basic periodic wave is a wave that has a motion that periodically repeats itself, or in other words, repeats its behavior every set period of time. Conceptually, one of the describing features of a periodic traversal wave is frequency. Frequency is the rate at which the wave completes a cycle, and is measured in Hertz (Hz = 1/sec)(7)
We can consider an imaginary **frequency space**, which represent the frequency at which a periodic traversal wave propagated at. As determined in the relationships, we learn that for a wave with a uniform speed (unchanged velocity), as the frequency in the frequency space increases, the wave length decreases, and vice versa. In other words, a wave's propagating periodic motion can be described either by its frequency.



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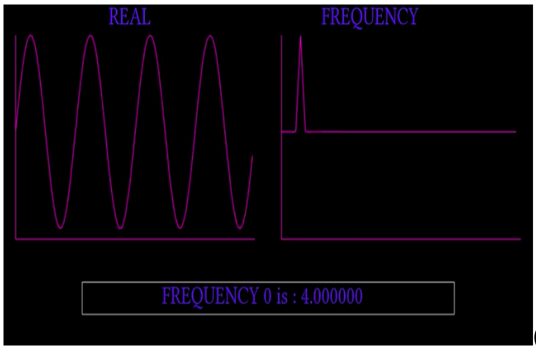


(4)

In modern uses, the application of wave analysis often requires the introduction of more than a single period wave. In other words, often waves are made of more than a single sin or cosine period motion and therefore are made of more than one frequency(6). The analysis of such waves becomes more complex than the traverse position equation of a simple traversal periodic wave (which is based on having only one sin wave)(4).

(4)

As mentioned, Fourier originally wanted to use the mathematical concept he developed to solve partial differential equations, specifically the heat equation (2). It was later discovered that the Fourier's Transformation can be used to evaluate the transformation from a wave's frequency space to a series of periodical waves that sum up to a final description of the complex wave's behavior. This transformation described a wave's motion and behavior in a form familiar to engineers for wave analysis and modification. (5)

3 Methodology and Mathematics I: Euler's Formula

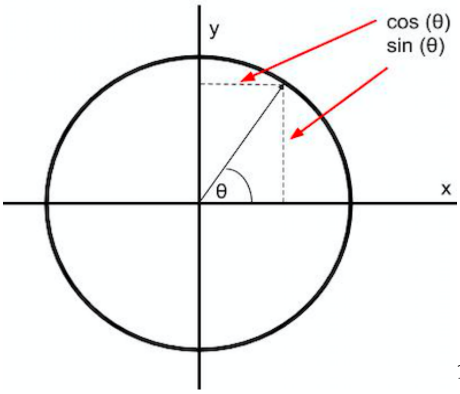
In this work, Fourier made a great use of a substitution well known to mathematicians:

$$e^{i\theta} = \cos\theta + i \cdot sin\theta$$

It was developed in the mid 1700's by a Swiss Mathematician, **Leonhard Euler** [pronounced 'Oiler'], who is well know to this day for his contributions for calculus mathematics.

3.1 Imaginary and Real Planes

Euler wanted to describe the motion of waves in imaginary planes. He started by considering the basic trigonometric and periodic functions of sines and cosines.

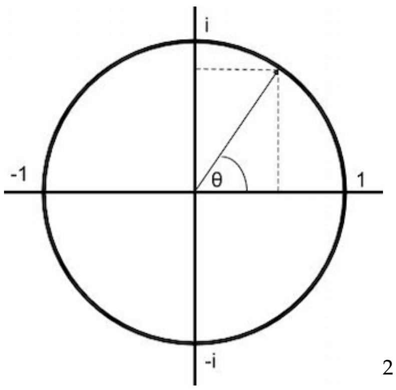


[1]

The image represents a simple periodical motion (circular motion) with a position described as a sinusoidal wave. By the trigonometric identity, we establish that: (12)

$$\cos^2\theta + \sin^2\theta = 1$$

The same application can be made on the plane of imaginary numbers.



[2]

Any value on the plane of imaginary numbers can be represented in the form

$$c = a + bi$$

We can define its conjugate value as

$$d = a - bi$$

[1]This figure was drawn on Google Drawings
[2]This figure was drawn on Google Drawings

Therefore, the length-squared of a complex number is given by the binomial identity (12)

$$c \cdot d = (a + bi)(a - bi) = a^2 + b^2$$

By comparing the length squared of a complex number to the length number given by the mentioned trigonometric identity for an x-y plane, we can consider the following transformation

Real Space	Imaginary Space
$\cos^2\theta + \sin^2\theta = 1$	a$^2 + b^2 = 1$
$\cos\theta$	a
$\sin\theta$	b

if we recall the value of an imaginary number,

$$c = a + bi$$

we conclude the following transformation for a value on the imaginary plane:

$$z = \cos\theta + i\sin\theta = x + iy$$

Using this transformation, we can redefine the following conjugates(12):

$$c = a + bi = \cos\theta + i \cdot \sin\theta$$

$$d = a - bi = \cos\theta - i \cdot \sin\theta$$

algebraically we can consider:

$\cos\theta + i\sin\theta$

$= \frac{1}{2} \cdot 2\cos\theta + \frac{1}{2} \cdot 2i\sin\theta$

$= \frac{1}{2}(\cos\theta + \cos\theta) + \frac{1}{2}(i\sin\theta + i\sin\theta)$

$= \frac{1}{2}(\cos\theta + \cos\theta - (i\sin\theta - i\sin\theta) + \frac{1}{2}(i\sin\theta + i\sin\theta + (\cos\theta - \cos\theta)$

$= \frac{1}{2}(\cos\theta - i\sin\theta + \cos\theta + i\sin\theta) + \frac{1}{2}(i\sin\theta + \cos\theta + i\sin\theta - \cos\theta)$

$= \frac{1}{2}((\cos\theta - i\sin\theta) + (\cos\theta + i\sin\theta)) + \frac{1}{2}((i\sin\theta + \cos\theta) - (\cos\theta - i\sin\theta))$

By looking at the trans formative conjugated that were redefined, **c and d** we can rewrite our equation(12):

$$\frac{1}{2}(d + c) + \frac{1}{2}(c - d) \qquad \text{(eqn. 1)}$$

Similarly, the following exponential function can be considered as a transformation between the real and imaginary spaces

$$
\begin{aligned}
&e^{i\theta} \\
&= \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{i\theta} \\
&= \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{i\theta} + (\frac{1}{2}e^{-i\theta} - \frac{1}{2}e^{-i\theta}) \\
&= (\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}) + (\frac{1}{2}e^{i\theta} - \frac{1}{2}e^{-i\theta}) \\
&= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) + \frac{1}{2}(e^{i\theta} - e^{-i\theta})
\end{aligned}
$$

notice the conjugates in the equations. Define them as(12):

$$m = e^{i\theta}, n = e^{-i\theta}$$

Therefore, with **m and n** defined, we can rewrite our equation(12):

$$\frac{1}{2}(m+n) + \frac{1}{2}(m-n) \qquad \text{(eqn. 2)}$$

By comparing the two equations (eqn. 1) and (eqn. 2), along with the comparison of the conjugate relationship, we can establish that **m = c** and therefore(12):

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad \text{(eqn. EF)}$$

This equation is also known as the Euler's Formula, and is heavily used in high-level math and engineering.

3.2 With Trigonometric Series

Interestingly enough, another method of proving Euler's formula, is by using trigonometric series (12).
Consider the first few terms of the following series (1):

$$\cos\theta = \sum_{n=0}^{N} \frac{(-1)^n \cdot \theta^{2n}}{(2n!)} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \qquad \text{(eqn. TS-1)}$$

$$\sin\theta = \sum_{n=0}^{N} \frac{(-1)^n \cdot \theta^{2n+1}}{(2n+1!)} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \qquad \text{(eqn. TS-2)}$$

$$e^x = \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} \qquad \text{(eqn. TS-3)}$$

Now, considering (eqn. TS-3), we can establish that:

$$e^{i\theta} = \sum_{n=0}^{N} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!}$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \frac{i^8\theta^8}{8!}$$

By considering the following chart of the values of imaginary numbers, we can substitute accordingly (1):

n	i^n	Value
1	i	$\sqrt{-1}$
2	i^2	-1
3	i^3	-$\sqrt{-1}$
4	i^4	1

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \frac{\theta^8}{8!}$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!}$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!})$$

We can now substitute the equations (eqn. TS-1) **and** (eqn. TS-2) which leads us to Euler's formula (1) (12)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad \text{(eqn. EF)}$$

4 Methodology and Mathematics II: Fourier's Transforms

In 1805, Joseph Fourier came up with the idea that "'any' function [f(X)] on the interval [0,1] can be written as a sum of sines and cosines" (2).
Recall from section —– the series of trigonometric periodic functions Fourier defined:

$$F(t) = s_n(t) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cdot cos(\omega \cdot t)) + b_n \cdot sin(\omega \cdot t)) \quad \text{(eqn. FS-1)}$$

To differentiate the Fourier transforms from the Fourier series, the amplitudes of the different frequencies (in the frequency space) are needed. This process is called **Fourier**

analysis and is greatly used in applications in the world of physics and engineering. **Fourier Analysis**, by its definition, is the "process of extracting from the signal the various frequencies and amplitudes that are present" (6). By using the **orthogonality property** of the periodical trigonometric functions of sines and cosines, the amplitudes **a and b** (also known as the Fourier coefficients) can be computed (5). The property defines that by taking a sine sand and a cosines functions (or two sines or two cosines), each a multiple of some fundamental frequency multiplying them together, and integrating that product over one period of the frequency, the results is always zero (5).

$$\int_{t=0}^{P} \cos(\omega t \cdot n) \cdot \cos(\omega t \cdot m) dt = 0 \qquad \text{(eqn. OP-1)}$$

and

$$\int_{t=0}^{P} \sin(\omega t \cdot n) \cdot \sin(\omega t \cdot m) dt = 0 \qquad \text{(eqn. OP-2)}$$

Unless m = +/- n, in which case:

$$\int_{t=0}^{P} \sin(\omega t \cdot n) \cdot \cos(\omega t \cdot m) dt = 0 \qquad \text{(eqn. OP-3)}$$

Note that both (eqn. OP-1) and (eqn. OP-2) equals to 1/(2f) if m = n. **Next**, we multiple and integrate F(t) (eqn. FS-1) over one period P, as follows (5):

$$\int_{t=0}^{P} F(t) \sin(\omega t \cdot n) dt$$

$$= \frac{a_0}{2} \int_{t=0}^{P} \sin(\omega t \cdot n) + \int_{t=0}^{P} \sum_{n=1}^{\infty} [a_n \cos(\omega t \cdot n) + b_n \sin(\omega t \cdot n)] \sin(\omega t \cdot m) dt$$

Such that all the terms of the summation vanish on integration besides

$$= \int_{t=0}^{P} b_m \sin^2(\omega t \cdot m) dt = b_m \int_{t=0}^{P} \sin^2(\omega t \cdot m) dt = \frac{b_m}{2f} = \frac{b_m \cdot P}{2}$$

Hence we conclude (5) (6):

$$\boxed{b_m = \frac{2}{P} \int_{t=0}^{P} F(t) \cdot \sin(\omega t \cdot m)}$$

And similarly:

$$\boxed{a_m = \frac{2}{P} \int_{t=0}^{P} F(t) \cdot \cos(\omega t \cdot m)}$$

Which represent the Fourier Coefficients in the Fourier Series.
By using the integrated definitions of Fourier's coefficients, the development of the Fourier transformation was possible. Considering the frequency space is an imaginary space used

for a convenient description of a periodical traversal wave's behavior, using the [Fourier] series he developed, Fourier defined two functions (5) (4):

- **f(x)** the function of the wave in real space.

$$f(t) = \int_{-\infty}^{\infty} F(K) \cdot e^{i \cdot 2\pi K t} dK = \int_{-\infty}^{\infty} F(K) \cdot e^{i\omega t} dK$$

- **F(K)** the function of the wave in frequency space.

$$F(K) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i \cdot 2\pi t K} dx = \int_{-\infty}^{\infty} F(K) \cdot e^{i\omega t} dx$$

Where K is the frequency of the traversal wave in cycles per second.
Note that the Fourier transforms in two dimensions (with the imaginary frequency space) is given by(6):

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i \cdot 2\pi (ux + vy)} dx dy$$

where **u and v** are spatial frequencies measures in the x and y directions respectively.
These equations represent the **transformation of a wave's periodical behavior** from a frequency space to real space, and vice versa (6). We see the use of imaginary numbers applies to the concept that the frequency space is a 'made-up' imaginary dimension. By the simple substitution provided by Euler's Formula (eqn. EF), we can see periodic motion in the space-transformative equations (6) (4):

$$f(t) = \int_{-\infty}^{\infty} F(K) \cdot (\cos(\omega t) + i \sin(\omega t)) dK$$

$$F(K) = \int_{-\infty}^{\infty} f(t) \cdot (\cos(-\omega t) + i \sin(-\omega t)) dx$$

Now, we can display these equations in a form known as **Discrete Fourier Transforms**(6), which essentially converts the integrals to summation form (4).

$$\boxed{f_n = \frac{1}{n} \sum_{K=0}^{N-1} F_K \cdot (\cos(\omega \frac{n}{N}) + i \sin(\omega \frac{n}{N}))} = \sum_{K=0}^{N-1} F_K \cdot e^{i \cdot \omega \frac{n}{N}}$$

$$\boxed{F_K = \sum_{n=0}^{N-1} f_n \cdot (\cos(-\omega \frac{n}{N}) + i \sin(-\omega \frac{n}{N}))} = \sum_{n=0}^{N-1} f_n \cdot e^{-i \cdot \omega \frac{n}{N}}$$

In this form, of summations of periodical motion, the behavior of any detected traversal disturbance in a medium can be displayed as a multiplication of matrices.

5 Problems and Fast Fourier Transformation

We have seen therefore that by using Euler's Formula as a substitution in Fourier's Series, any detected traversal wave can be displayed and presented as the product (multiplication) of multiple matrices. As programmers and engineers attempted to automate this process, a problem was encountered: it often took a great amount of time to calculate the result of the matrix multiplication, which significantly slowed down the process of translating the wave into matrices and then convert those into readable information and data.
To resolve this issue, mathematicians developed algorithms and methods to quicken the process of calculating the Fourier transforms. The most common and main two algorithms used today are the Cooley - Tukey Algorithm(11), and the Butterfly Diagram.

5.1 The Cooley - Tukey Algorithm

In 1965, two Mathematicians, Cooley and Tukey, published a paper about mathematical methods initially introduced by Gauss in 1805(8), which allowed quick computation of the **DFT** [Discrete Fourier transforms], which is mostly done by programs and code. To understand the Cooley-Tukey algorithm, the proper way in which the Fourier Transforms is calculated by a computer program must be understood: represented as an element of an array - a list of items in computer programs. By referring to the arrays: the Discrete Fourier

Array	Space	Notation
Array One	Real Space	X[f]
Array Two	Frequency Space	X[n]

Transformation can be represented as the following formula (10):

$$X[f] = \sum_{n=0}^{N-1} X[n] e^{-i\omega \frac{n}{N}}$$

An efficiency computer program to build such an array has an order of $O(N^2)$ Which means the relationship between the amount of elements to add to the array to the time it will take (in repetitions of the algorithm), is quadratic, and therefore not considerably truly efficient, especially considering the applies matrices and series are incredibly large(13). The algorithm is built on the concept of exploiting the symmetry of the term: $e^{-i\omega \frac{n}{N}} = e^{-i\frac{2\pi f n}{N}}$ Where we define the following (10):

$$W_N = e^{-i\frac{2\pi}{N}} \iff W_N^{fn} = e^{-i\frac{2\pi f \cdot n}{N}} = e^{-i\omega \frac{n}{N}}$$

The symmetry of the term is special in its complex conjugate symmetry (see section 3.1) as well as in its periodicity both in **n** (number of cycles) and **f**(element's frequency) (10). The development of the algorithm begins by sorting the elements into odd and even indexed sub-sequences (13):

$$X[f] = \sum_{n=0}^{N-1} X[n]W_N^{fn} = \sum_{n=odd} X[n]W_N^{fn} + \sum_{n=even} X[n]W_N^{fn}$$

By referring to the definition of odd and even integers (13):

Term	Definition
Set Z	Set of all integers $\{0, \pm 1, \pm 2, \pm 3, \cdots\}$
Even Integers	$\forall \ (2r) \in Z \quad \mid \quad r \in Z$
Odd Integers	$\forall \ (2r+1) \in Z \quad \mid \quad r \in Z$

we define (13) (10):

$$X[f] = \sum_{n=0}^{\frac{N}{2}-1} X[2r]W_N^{f2r} + \sum_{n=0}^{\frac{N}{2}-1} X[2r+1]W_N^{f(2r+1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} X[2r](W_N^2)^{fr} + W_N^f \sum_{n=0}^{\frac{N}{2}-1} X[2r+1](W_N^2)^{fr}$$

$$= \boxed{\sum_{n=0}^{\frac{N}{2}-1} X[2r]W_{\frac{N}{2}}^{fr} + W_N^f \sum_{n=0}^{\frac{N}{2}-1} X[2r+1]W_{\frac{N}{2}}^{fr}}$$

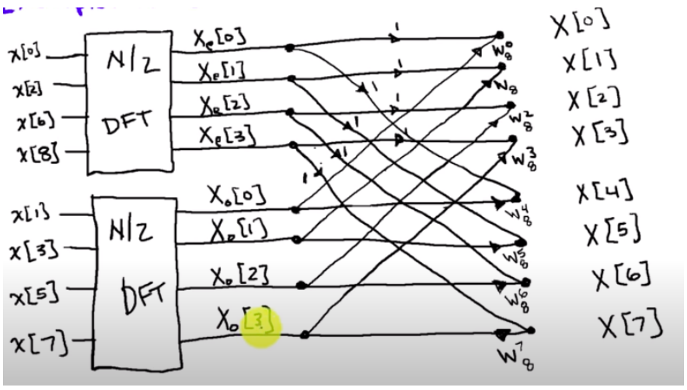
The result is an array X[f] of DFT split into two arrays of DFT: odd integers, and event integers. Which can also be represented as:

$$X[f] = X_{even}[f] + W_N^f \cdot X_{odd}[f]$$

Each one of the size N/2.

5.2 Butterfly Diagrams

In the Cooley-Tukey algorithm, the single array (or rather single column matrix) of elements from Fourier's series was split into two arrays of Discrete Fourier Trams forms (even and odd integers). This set up allows the computing program to avoid any matrix multiplications, and significantly quicker, compute the desired product. This is done by graphing **butterfly diagrams**(11).



(10)

As seen in the attached figure, the two arrays are arranged in a single column. By creating diagonal lines (which give the look of a butterfly to the diagram), they become organized in the desired form, which allows us to avoid any matrix computation(11). However, the butterfly diagram in the figure only describes an array that was split once. When arrays are on a much larger scale, division into two arrays only one times often will make a difference not significant enough to slow down the computations. Therefore, we can continue and divide each $\frac{N}{2}$ DFT by two, until we get as many **arrays of P elements** each.

$$\frac{N}{2}, \frac{N}{4}, \frac{N}{8}, \cdots, \frac{N}{2^{m-1}}, \frac{N}{2^m}, 1$$

The 'cost' or time)in algorithm repetition per element) for N elements is therefore(10):

$$\frac{N}{2^p} = 1 \longrightarrow \frac{N^2}{2^P} + P \cdot N = \frac{N^2}{N} + N \log_2 N$$

Therefore, since $N \ll N \log_2 N$, the 'cost' for an N large array is(10)

$$\boxed{O(N \log_2 N)}$$

5.3 Algorithm Comparison

With the Cooley-Tukey algorithm, the computer program builds the desired array with an order of $O(\log_2(N))$, which is significantly more effective. Therefore, there is no doubt,

Number of Elements	FFT (Cooley - Tukey)	DFT	Ratio (FFT/DFT)
N	O(N^2)	O(N$\cdot \log_2(N)$)	O($\log_2(N)/N$)
1	1	0	0
10	100	34	3.4
100	10000	664	0.0664
1000	1000000	9966	0.009966
10000	100000000	132878	0.00132878
100000	10000000000	1660965	$1.660965 \cdot 10^{-4}$

that especially when the number of elements is getting bigger (which if often the case), the Fast Fourier Transform takes significantly much less time than the Discrete Fourier Transform.

6 Applications and Use

Essentially, as can be concluded from its derivation, the Fourier transforms can be used to calculate any transformation from an imaginary space to real space, or evaluate it over a finite domain.
First of all, the method of Fourier transforms is used for its original purpose: solve partial differential equations and initial value problems of such kind. A great example is the wave equation (eqn. W)(7), which can be solved by using Fourier transforms. We can multiply both sides by a factor of Euler's formula (eqn. EF) and integrate with respect to x (horizontal displacement)(3):

$$v^2 \frac{\partial^2 y}{\partial x^2} \cdot e^{-ikx} = \frac{\partial^2 y}{\partial t^2}(x,t) \cdot e^{-ikx}$$

$$\int_{-\infty}^{\infty} v^2 \frac{\partial^2 y}{\partial x^2} \cdot e^{-ikx} dx = \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2}(x,t) \cdot e^{-ikx} dx$$

From section 4, we establish the following Fourier transform(3):

$$\int_{-\infty}^{\infty} f'(x) \cdot e^{-i \cdot Kx} dx = iK \int_{-\infty}^{\infty} f(x) \cdot e^{-i \cdot Kx} dx = iKF(K)$$

for a uniform (unchanged) velocity v, we can integrate by parts as follows (3):

$$u = e^{-iKx}, dv = f'(x)dx$$

such that, by the Fourier transforms:

$$\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2}(x,t) \cdot e^{-ikx} dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(x,t) \cdot e^{-ikx} dx = \frac{\partial^2}{\partial t^2} Y(K,t)$$

Hence(3):

$$\boxed{\frac{\partial^2}{\partial t2} Y(K,t) = -v^2 k^2 Y(K,t)}$$

Although not a complete solution to the wave equation, the Fourier transforms had allowed us to significantly simplify the wave equation to a form of two differential equation that equate to each other.
Another example, is the Heat Equation (see introduction 1), which, to remind, solving it was the main objective of the Fourier Series. It was solves by representing the general solution (9) of the 'traversal heat disturbances in a medium by

$$T(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

Which (clearly) is **related to a DFT** (Discrete Fourier Transform)
Furthermore, The most commonly known practical application of Fourier transforms is among electrical engineers for the **translation of sound waves into isolated traversal**

waves and words in a readable form(8). To engineers, it is best known as digital signal processing. It includes the detection of sound over a time interval, and its isolation from background-disturbing sounds. This is done by the Fourier transforms, which translates the total sound waves into a sum of periodical functions, which is a form that allows us to simply cancel or neglect the 'disturbing waves' that are unnecessary for the digital processing. This isolated wave in the form of a sum of periodical trigonometric functions is then converted to the form of discrete Fourier transforms, and by matrix multiplication (which is relatively slow), is processed into a user-readable form. This application is used in many modern technologies, including voice recognition, earthquakes detection, and electromagnetic radiation detection (8).

7 Summary

All in all, there is no doubt the Fourier transformation is a significant milestone in the development of modern mathematics. Using Euler's Formula that allows convenient computing of imaginary numbers in the form of periodical trigonometric functions, the Fourier transformation allows us to transform wave equations and behavioral formulas of periodical motions from an imaginary space to a real space by using a series of sinusoidal waves in a form of a sum, which can also be represented as an integral over a finite domain. The Fourier transformation,as explored, can then be displayed in its discrete form, which can be evaluated by the multiplication of multiple matrices In conclusion, the Fourier transformation is prohibitively too slow to translate traversal waves into data of readable form, considering matrix multiplication of a large scale often requires a great amount of computing. In order to resolve this issue, engineers have developed the Cooley-Tukey algorithm and the butterfly diagram which allowed a fast Fourier transformation, and therefore a quick and reliable translation of detected disturbances in a medium (wave) into readable and usable data. to summarize, the Fourier transforms are extremely helpful and useful in the development of modern applications such as radiation detection, data integration over distance, voice recognition, and more.

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